Open Colorings

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Definitions

Definition

Let X be a topological space. $K \subseteq [X]^2$ is open if the set $\{(x, y) \in X^2 : x, y \in K\}$ is open in X^2. A partition $[X]^2 = K_0 \cup K_1$ such that K_0 is open is called open partition.

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$OGA(X)$ is the following sentence:
For every open partition $[X]^2 = K_0 \cup K_1$, either:

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2. *There is a family $\langle H_n : n \in \omega \rangle$ such that $X = \bigcup_{n \in \omega} H_n$ and $[H_n]^2 \subseteq K_1$ for every $n \in \omega$ (σ-1-homogeneous).*

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For $OGA^*(X)$ it means H has size $\mathfrak{c}$ in the option 1.

Todorčević's conjecture:

Is it consitent with ZFC:

If X is a regular space with no uncountable discrete subspace and if $[X]^2 = K_0 \cup K_1$ is a given partition with K_0 open, then either there is an uncountable 0-homogeneous set, or else X is the union of countably many 1-homogeneous sets.

ZFC facts

Theorem (Todorčević)

$OGA^*(\omega^\omega)$ is true.

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Let X, Y be topological space such that Y is T_2, and there is a continuos function f from X onto Y; then $OGA(X)$ implies $OGA(Y)$.

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Definition

Let X be a topological space and $A, B \subseteq X$, we define the set

$$A \otimes B = \{\{x, y\} \in [X]^2 : x \in A \wedge y \in B \wedge x \neq y\}.$$

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2. OGA(ω_1) with the discrete topology is false.
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4. Double arrow space.

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X is strong Choquet if player II has a win strategy in the strong Choquet game.

Proof

Fix an open partition $[X]^2 = K_0 \cup K_1$ and assume that X can not be covered by countably many 1-homogeneous sets. Let σ be a winining strategy for II in the strong Choquet game G_X.

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We will define for $s \in \{0,1\}^{<\omega}$ U_s such that $\langle U_s : s \in \{0,1\}^{<\omega} \rangle$ is a Cantor schemme with the following properties:

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2. if $s, t \in \{0,1\}^{<\omega}$ and $t \supseteq s$, then $U_s \supseteq U_t$,
3. if $s \in 2^{<\omega}$, then $U_{s^\frown 0} \cap U_{s^\frown 1} = \emptyset$, $U_{s^\frown 0} \otimes U_{s^\frown 1} \subseteq K_0$, and

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4. U_s can not be covered by countably many 1-homogeneous sets, for every $s \in \{0,1\}^{<\omega}$.

Proof:

Claim:

Let U be an open set such that can not be covered by countably many 1-homogeneous set, then there exist $x_0, x_1 \in U$, and $U^0, U^1 \subseteq U$ disjoint such that:

1. $\forall i \in \{0,1\}(x_i \in U^i)$,
2. $U^0 \otimes U^1 \subseteq K_0$, and
3. $\forall i \in \{0,1\}$ U^i can not be covered by countably many 1-homogeneous set.

proof

Let $U_\emptyset = X$, use the previous claim, and find x_0, x_1, U^0, U^1 acording to the conclusion of the claim.

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Let $F : 2^\omega \longrightarrow X$ be such that $F(f) \in \bigcap_{n\in\omega} U_{f\restriction n}$, and $F''[2^\omega]$ is a 0-homogeneous set of size $\mathfrak{c}$.

PFA part

Theorem (Todorčević)

PFA implies OGA.

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Theorem

OGA is aquivalent with:
$OGA(X)$ holds for every subespace X of the Sorgenfrey line.

Proof

Let $X \subset \mathbb{S}$ be uncountable, $[X]^2 = K_0 \cup K_1$ some open partion, and $\mathcal{B} = \{B_n : n \in \omega\}$ a countable basis of X acording to the usual topology.

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- Let τ_0 the topoloy generated by $\{A_n, B_n : n \in \omega\}$.

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- There exists $H \subseteq X$ an uncountable set such that $[H]^2 \subseteq K_0'$.
- Then $[H]^2 \subseteq K_0' \subseteq K_0$.

Proof

Suppose $X = \bigcup_{n\in\omega} X_n$ with $[X_n]^2 \cap K_0 = \emptyset$, then WLOG X_n does not have isolated points or X_n is a single point.

- Suposse $\{x, y\} \in K_0 \cap [X_k]^2$, for some $k \in \omega$.

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- $\exists t \in B_{n_0} \subseteq [y, y+\epsilon)_X \cap X_k$.
- $x \in A_{n_0}$, then $\{t, y\} \in K_0' \cap [X_k]^2$ contradiction.

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Theorem (Todorčević,(PFA))

If X is a regular space, then one of the following holds for X:

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2. *X is an hereditary Lindelöf space.*

What about the L space

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Question

Is there a regular L-space such that OGA(L)?

Big questions

1. (Todorčević) ¿Does OGA imply $2^{\aleph_0} = \aleph_2$?

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2. (Todorčević) ¿Is OGA consistent with $add(\mathcal{N}) = \aleph_1$?

Thank You!